Utah State Office of Education

core standards *for* MATHEMATICS

Revised October 2012

The Utah State Board of Education, in January of 1984, established policy requiring the identification of specific core standards to be met by all K–12 students in order to graduate from Utah's secondary schools. The Utah State Board of Education regularly updates the Utah Core Standards, while parents, teachers, and local school boards continue to control the curriculum choices that reflect local values.

The Utah Core Standards are aligned to scientifically based content standards. They drive high quality instruction through statewide comprehensive expectations for all students. The standards outline essential knowledge, concepts, and skills to be mastered at each grade level or within a critical content area. The standards provide a foundation for ensuring learning within the classroom.

Utah STATE OFFICE of Education

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Utah STATE OFFICE of Education

UTAH CORE STATE STANDARDS *for* MATHEMATICS

Adopted August 2010

by the

Utah State Board of Education

Revised October 2012

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Introduction

Toward greater focus and coherence

> *Mathematics experiences in early childhood settings should concentrate on (1) number (which includes whole number, operations, and relations) and (2) geometry, spatial relations, and measurement, with more mathematics learning time devoted to number than to other topics. Mathematical process goals should be integrated in these content areas.*
>
> —Mathematics Learning in Early Childhood, National Research Council, 2009

> *The composite standards [of Hong Kong, Korea and Singapore] have a number of features that can inform an international benchmarking process for the development of K–6 mathematics standards in the U.S. First, the composite standards concentrate the early learning of mathematics on the number, measurement, and geometry strands with less emphasis on data analysis and little exposure to algebra. The Hong Kong standards for grades 1–3 devote approximately half the targeted time to numbers and almost all the time remaining to geometry and measurement.*
>
> —Ginsburg, Leinwand and Decker, 2009

> *Because the mathematics concepts in [U.S.] textbooks are often weak, the presentation becomes more mechanical than is ideal. We looked at both traditional and non-traditional textbooks used in the US and found this conceptual weakness in both.*
>
> —Ginsburg et al., 2005

> *There are many ways to organize curricula. The challenge, now rarely met, is to avoid those that distort mathematics and turn off students.*
>
> —Steen, 2007

For over a decade, research studies of mathematics education in high-performing countries have pointed to the conclusion that the mathematics curriculum in the United States must become substantially more focused and coherent in order to improve mathematics achievement in this country. To deliver on the promise of common standards, the standards must address the problem of a curriculum that is "a mile wide and an inch deep." These Standards are a substantial answer to that challenge.

It is important to recognize that "fewer standards" are no substitute for focused standards. Achieving "fewer standards" would be easy to do by resorting to broad, general statements. Instead, these Standards aim for clarity and specificity.

Assessing the coherence of a set of standards is more difficult than assessing their focus. William Schmidt and Richard Houang (2002) have said that content standards and curricula are coherent if they are:

> *articulated over time as a sequence of topics and performances that are logical and reflect, where appropriate, the sequential or hierarchical nature of the disciplinary content from which the subject matter derives. That is, what and how students are taught should reflect not only the topics that fall within a certain academic discipline,* ***but also the key ideas*** *that determine how knowledge is organized and generated within that discipline. This implies that to be*

> *coherent, a set of content standards must evolve from particulars (e.g., the meaning and operations of whole numbers, including simple math facts and routine computational procedures associated with whole numbers and fractions) to deeper structures inherent in the discipline. These deeper structures then serve as a means for connecting the particulars (such as an understanding of the rational number system and its properties). (emphasis added)*

These Standards endeavor to follow such a design, not only by stressing conceptual understanding of key ideas, but also by continually returning to organizing principles such as place value or the properties of operations to structure those ideas.

In addition, the "sequence of topics and performances" that is outlined in a body of mathematics standards must also respect what is known about how students learn. As Confrey (2007) points out, developing "sequenced obstacles and challenges for students…absent the insights about meaning that derive from careful study of learning, would be unfortunate and unwise." In recognition of this, the development of these Standards began with research-based learning progressions detailing what is known today about how students' mathematical knowledge, skill, and understanding develop over time.

Understanding mathematics

These Standards define what students should understand and be able to do in their study of mathematics. Asking a student to understand something means asking a teacher to assess whether the student has understood it. But what does mathematical understanding look like? One hallmark of mathematical understanding is the ability to justify, in a way appropriate to the student's mathematical maturity, why a particular mathematical statement is true or where a mathematical rule comes from. There is a world of difference between a student who can summon a mnemonic device to expand a product such as $(a + b)(x + y)$ and a student who can explain where the mnemonic comes from. The student who can explain the rule understands the mathematics, and may have a better chance to succeed at a less familiar task such as expanding $(a + b + c)(x + y)$. Mathematical understanding and procedural skill are equally important, and both are assessable using mathematical tasks of sufficient richness.

The Standards set grade-specific standards but do not define the intervention methods or materials necessary to support students who are well below or well above grade-level expectations. It is also beyond the scope of the Standards to define the full range of supports appropriate for English language learners and for students with special needs. At the same time, all students must have the opportunity to learn and meet the same high standards if they are to access the knowledge and skills necessary in their post-school lives. The Standards should be read as allowing for the widest possible range of students to participate fully from the outset, along with appropriate accommodations to ensure maximum participation of students with special education needs. For example, for students with disabilities reading should allow for use of Braille, screen reader technology, or other assistive devices, while writing should include the use of a scribe, computer, or speech-to-text technology. In a similar vein, speaking and listening should be interpreted broadly to include sign language. No set of grade-specific standards can fully reflect the great variety in abilities, needs, learning rates, and achievement levels of students in any given classroom. However, the Standards do provide clear signposts along the way to the goal of college and career readiness for all students.

The Standards begin on page 6 with eight Standards for Mathematical Practice.

How to read the grade level standards

Standards define what students should understand and be able to do.

Clusters are groups of related standards. Note that standards from different clusters may sometimes be closely related, because mathematics is a connected subject.

Domains are larger groups of related standards. Standards from different domains may sometimes be closely related.

Domain

Number and Operations in Base Ten **3.NBT**

Use place value understanding and properties of operations to perform multi-digit arithmetic.

Standard

1. Use place value understanding to round whole numbers to the nearest 10 or 100.
2. Fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction.
3. Multiply one-digit whole numbers by multiples of 10 in the range 10–90 (e.g., 9 x 80, 5 x 60) using strategies based on place value and properties of operations.

Cluster

These Standards do not dictate curriculum or teaching methods. For example, just because topic A appears before topic B in the standards for a given grade, it does not necessarily mean that topic A must be taught before topic B. A teacher might prefer to teach topic B before topic A, or might choose to highlight connections by teaching topic A and topic B at the same time. Or, a teacher might prefer to teach a topic of his or her own choosing that leads, as a byproduct, to students reaching the standards for topics A and B.

What students can learn at any particular grade level depends upon what they have learned before. Ideally then, each standard in this document might have been phrased in the form, "Students who already know … should next come to learn …." But at present this approach is unrealistic—not least because existing education research cannot specify all such learning pathways. Of necessity therefore, grade placements for specific topics have been made on the basis of state and international comparisons and the collective experience and collective professional judgment of educators, researchers and mathematicians. One promise of common state standards is that over time they will allow research on learning progressions to inform and improve the design of standards to a much greater extent than is possible today. Learning opportunities will continue to vary across schools and school systems, and educators should make every effort to meet the needs of individual students based on their current understanding.

These Standards are not intended to be new names for old ways of doing business. They are a call to take the next step. It is time for states to work together to build on lessons learned from two decades of standards based reforms. It is time to recognize that standards are not just promises to our children, but promises we intend to keep.

Mathematics | Standards for Mathematical Practice

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up:* adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy).

1 Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2 Reason abstractly and quantitatively.

Mathematically proficient students make sense of the quantities and their relationships in problem situations. Students bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

3 Construct viable arguments and critique the reasoning of others.

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two

plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4 Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5 Use appropriate tools strategically.

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6 Attend to precision.

Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7 Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

8 Look for and express regularity in repeated reasoning.

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Connecting the Standards for Mathematical Practice to the Standards for Mathematical Content

The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.

The Standards for Mathematical Content are a balanced combination of procedure and understanding. Expectations that begin with the word “understand” are often especially good opportunities to connect the practices to the content. Students who lack understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, they may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an overview, or deviate from a known procedure to find a shortcut. In short, a lack of understanding effectively prevents a student from engaging in the mathematical practices.

In this respect, those content standards which set an expectation of understanding are potential “points of intersection” between the Standards for Mathematical Content and the Standards for Mathematical Practice. These points of intersection are intended to be weighted toward central and generative concepts in the school mathematics curriculum that most merit the time, resources, innovative energies, and focus necessary to qualitatively improve the curriculum, instruction, assessment, professional development, and student achievement in mathematics.

Utah STATE OFFICE of Education

UTAH CORE STATE STANDARDS *for* MATHEMATICS

HIGH SCHOOL

Mathematics Standards for High School

THE HIGH SCHOOL STANDARDS specify the mathematics that all students should study in order to be college and career ready. Additional mathematics that students should learn in order to take advanced courses such as calculus, advanced statistics, or discrete mathematics is indicated by **(+)**, as in this example:

> (+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers).

All standards without a (+) symbol should be in the common mathematics curriculum for all college and career ready students. Standards with a (+) symbol may also appear in courses intended for all students.

The high school standards are listed in **CONCEPTUAL CATEGORIES**:

- Number and Quantity
- Algebra
- Functions
- Modeling
- Geometry
- Statistics and Probability

CONCEPTUAL CATEGORIES portray a coherent view of high school mathematics; a student's work with functions, for example, crosses a number of traditional course boundaries, potentially up through and including calculus.

MODELING is best interpreted not as a collection of isolated topics but in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific **modeling standards** appear throughout the high school standards indicated by a star symbol (★). The star symbol sometimes appears on the heading for a group of standards; in that case, it should be understood to apply to all standards in that group.

Mathematics | High School—Number and Quantity

NUMBERS AND NUMBER SYSTEMS. During the years from kindergarten to eighth grade, students must repeatedly extend their conception of number. At first, "number" means "counting number": 1, 2, 3…. Soon after that, 0 is used to represent "none" and the whole numbers are formed by the counting numbers together with zero. The next extension is fractions. At first, fractions are barely numbers and tied strongly to pictorial representations. Yet by the time students understand division of fractions, they have a strong concept of fractions as numbers and have connected them, via their decimal representations, with the base-ten system used to represent the whole numbers. During middle school, fractions are augmented by negative fractions to form the rational numbers. In Grade 8, students extend this system once more, augmenting the rational numbers with the irrational numbers to form the real numbers. In high school, students will be exposed to yet another extension of number, when the real numbers are augmented by the imaginary numbers to form the complex numbers.

With each extension of number, the meanings of addition, subtraction, multiplication, and division are extended. In each new number system—integers, rational numbers, real numbers, and complex numbers—the four operations stay the same in two important ways: They have the commutative, associative, and distributive properties and their new meanings are consistent with their previous meanings.

Extending the properties of whole-number exponents leads to new and productive notation. For example, properties of whole-number exponents suggest that $(5^{1/3})^3$ should be $5^{(1/3)\cdot 3} = 5^1 = 5$ and that $5^{1/3}$ should be the cube root of 5.

Calculators, spreadsheets, and computer algebra systems can provide ways for students to become better acquainted with these new number systems and their notation. They can be used to generate data for numerical experiments, to help understand the workings of matrix, vector, and complex number algebra, and to experiment with non-integer exponents.

QUANTITIES. In real world problems, the answers are usually not numbers but quantities: numbers with units, which involves measurement. In their work in measurement up through Grade 8, students primarily measure commonly used attributes such as length, area, and volume. In high school, students encounter a wider variety of units in modeling, e.g., acceleration, currency conversions, derived quantities such as person-hours and heating degree days, social science rates such as per-capita income, and rates in everyday life such as points scored per game or batting averages. They also encounter novel situations in which they themselves must conceive the attributes of interest. For example, to find a good measure of overall highway safety, they might propose measures such as fatalities per year, fatalities per year per driver, or fatalities per vehicle-mile traveled. Such a conceptual process is sometimes called quantification. Quantification is important for science, as when surface area suddenly "stands out" as an important variable in evaporation. Quantification is also important for companies, which must conceptualize relevant attributes and create or choose suitable measures for them.

Number and Quantity Overview

The Real Number System

- **Extend the properties of exponents to rational exponents.**
- **Use properties of rational and irrational numbers.**

Quantities

- **Reason quantitatively and use units to solve problems.**

The Complex Number System

- **Perform arithmetic operations with complex numbers.**
- **Represent complex numbers and their operations on the complex plane.**
- **Use complex numbers in polynomial identities and equations.**

Vector and Matrix Quantities

- **Represent and model with vector quantities.**
- **Perform operations on vectors.**
- **Perform operations on matrices and use matrices in applications.**

MATHEMATICAL PRACTICES

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

The Real Number System N-RN

Extend the properties of exponents to rational exponents.

1. Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})^3 = 5(^{1/3})^3$ to hold, so $(5^{1/3})^3$ must equal 5.*
2. Rewrite expressions involving radicals and rational exponents using the properties of exponents.

Use properties of rational and irrational numbers.

3. Explain why the sum or product of two rational numbers are rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

Quantities★ N-Q

Reason quantitatively and use units to solve problems.

1. Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.
2. Define appropriate quantities for the purpose of descriptive modeling.
3. Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.

The Complex Number System N-CN

Perform arithmetic operations with complex numbers.

1. Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.
2. Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.
3. **(+)** Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers.

Represent complex numbers and their operations on the complex plane.

4. **(+)** Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.
5. **(+)** Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation. *For example, $(-1 + \sqrt{3}\, i)^3 = 8$ because $(-1 + \sqrt{3}\, i)$ has modulus 2 and argument 120°.*
6. **(+)** Calculate the distance between numbers in the complex plane as the modulus of the difference, and the midpoint of a segment as the average of the numbers at its endpoints.

Use complex numbers in polynomial identities and equations.

7. Solve quadratic equations with real coefficients that have complex solutions.
8. **(+)** Extend polynomial identities to the complex numbers. *For example, rewrite $x^2 + 4$ as $(x + 2i)(x - 2i)$.*
9. **(+)** Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

Vector and Matrix Quantities N-VM

Represent and model with vector quantities.

1. **(+)** Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., $\boldsymbol{v}$, $|\boldsymbol{v}|$, $||\boldsymbol{v}||$, v).
2. **(+)** Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.
3. **(+)** Solve problems involving velocity and other quantities that can be represented by vectors.

Perform operations on vectors.

4. **(+)** Add and subtract vectors.
   - **a.** Add vectors end-to-end, component-wise, and by the parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes.
   - **b.** Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.
   - **c.** Understand vector subtraction $\boldsymbol{v} - \boldsymbol{w}$ as $\boldsymbol{v} + (-\boldsymbol{w})$, where $-\boldsymbol{w}$ is the additive inverse of $\boldsymbol{w}$, with the same magnitude as $\boldsymbol{w}$ and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order, and perform vector subtraction component-wise.
5. **(+)** Multiply a vector by a scalar.
   - **a.** Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as $c(v_x, v_y) = (cv_x, cv_y)$.
   - **b.** Compute the magnitude of a scalar multiple $c\boldsymbol{v}$ using $||c\boldsymbol{v}|| = |c|\boldsymbol{v}$. Compute the direction of $c\boldsymbol{v}$ knowing that when $|c|\boldsymbol{v} \neq 0$, the direction of $c\boldsymbol{v}$ is either along $\boldsymbol{v}$ (for $c > 0$) or against $\boldsymbol{vs}$ (for $c < 0$).

Perform operations on matrices and use matrices in applications.

6. **(+)** Use matrices to represent and manipulate data, e.g., to represent payoffs or incidence relationships in a network.
7. **(+)** Multiply matrices by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled.
8. **(+)** Add, subtract, and multiply matrices of appropriate dimensions.
9. **(+)** Understand that, unlike multiplication of numbers, matrix multiplication for square matrices is not a commutative operation, but still satisfies the associative and distributive properties.
10. **(+)** Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.
11. **(+)** Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.
12. **(+)** Work with 2 × 2 matrices as transformations of the plane, and interpret the absolute value of the determinant in terms of area.

Mathematics | High School—Algebra

EXPRESSIONS. An expression is a record of a computation with numbers, symbols that represent numbers, arithmetic operations, exponentiation, and, at more advanced levels, the operation of evaluating a function. Conventions about the use of parentheses and the order of operations assure that each expression is unambiguous. Creating an expression that describes a computation involving a general quantity requires the ability to express the computation in general terms, abstracting from specific instances.

Reading an expression with comprehension involves analysis of its underlying structure. This may suggest a dif-ferent but equivalent way of writing the expression that exhibits some different aspect of its meaning. For example, $p + 0.05p$ can be interpreted as the addition of a 5% tax to a price p. Rewriting p + $0.05p$ as $1.05p$ shows that adding a tax is the same as multiplying the price by a constant factor.

Algebraic manipulations are governed by the properties of operations and exponents, and the conventions of algebraic notation. At times, an expression is the result of applying operations to simpler expressions. For example, $p + 0.05p$ is the sum of the simpler expressions p and $0.05p$. Viewing an expression as the result of operation on simpler expressions can sometimes clarify its underlying structure.

A spreadsheet or a computer algebra system (CAS) can be used to experiment with algebraic expressions, per-form complicated algebraic manipulations, and understand how algebraic manipulations behave.

EQUATIONS AND INEQUALITIES. An equation is a statement of equality between two expressions, often viewed as a question asking for which values of the variables the expressions on either side are in fact equal. These values are the solutions to the equation. An identity, in contrast, is true for all values of the variables; identities are often developed by rewriting an expression in an equivalent form.

The solutions of an equation in one variable form a set of numbers; the solutions of an equation in two variables form a set of ordered pairs of numbers, which can be plotted in the coordinate plane. Two or more equations and/or inequalities form a system. A solution for such a system must satisfy every equation and inequality in the system.

An equation can often be solved by successively deducing from it one or more simpler equations. For example, one can add the same constant to both sides without changing the solutions, but squaring both sides might lead to extraneous solutions. Strategic competence in solving includes looking ahead for productive manipulations and anticipating the nature and number of solutions.

Some equations have no solutions in a given number system, but have a solution in a larger system. For example, the solution of $x + 1 = 0$ is an integer, not a whole number; the solution of $2x + 1 = 0$ is a rational number, not an

integer; the solutions of $x^2 - 2 = 0$ are real numbers, not rational numbers; and the solutions of $x^2 + 2 = 0$ are complex numbers, not real numbers.

The same solution techniques used to solve equations can be used to rearrange formulas. For example, the formula for the area of a trapezoid, $A = ((b_1+b_2)/2)h$, can be solved for h using the same deductive process.

Inequalities can be solved by reasoning about the properties of inequality. Many, but not all, of the properties of equality continue to hold for inequalities and can be useful in solving them.

CONNECTIONS TO FUNCTIONS AND MODELING. Expressions can define functions, and equivalent expressions define the same function. Asking when two functions have the same value for the same input leads to an equation; graphing the two functions allows for finding approximate solutions of the equation. Converting a verbal description to an equation, inequality, or system of these is an essential skill in modeling.

Algebra Overview

Seeing Structure in Expressions

- **Interpret the structure of expressions.**
- **Write expressions in equivalent forms to solve problems.**

Arithmetic with Polynomials and Rational Expressions

- **Perform arithmetic operations on polynomials.**
- **Understand the relationship between zeros and factors of polynomials.**
- **Use polynomial identities to solve problems.**
- **Rewrite rational expressions.**

Creating Equations

- **Create equations that describe numbers or relationships.**

Reasoning with Equations and Inequalities

- **Understand solving equations as a process of reasoning and explain the reasoning.**
- **Solve equations and in-equalities in one variable.**
- **Solve systems of equations.**
- **Represent and solve equations and inequalities graphically.**

MATHEMATICAL PRACTICES

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Seeing Structure in Expressions A-SSE

Interpret the structure of expressions.

1. Interpret expressions that represent a quantity in terms of its context.★

- **a.** Interpret parts of an expression, such as terms, factors, and coefficients.
- **b.** Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1+r)^n$ *as the product of P and a factor not depending on P.*

2. Use the structure of an expression to identify ways to rewrite it. *For example, see* $x^4 - y^4$ *as* $(x^2)^2 - (y^2)^2$*, thus recognizing it as a difference of squares that can be factored as* $(x^2 - y^2)(x^2 + y^2)$.

Write expressions in equivalent forms to solve problems.

3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★

- **a.** Factor a quadratic expression to reveal the zeros of the function it defines.
- **b.** Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.
- **c.** Use the properties of exponents to transform expressions for exponential functions. *For example the expression* 1.15^t *can be rewritten as* $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*

4. Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.*★

Arithmetic with Polynomials and Rational Expressions A-APR

Perform arithmetic operations on polynomials.

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Understand the relationship between zeros and factors of polynomials.

2. Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.

3. Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

Use polynomial identities to solve problems.

4. Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity* $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)v$ *can be used to generate Pythagorean triples.*

5. **(+)** Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.[28]

Rewrite rational expressions.

6. Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x) +$

[28] The Binomial Theorem can be proved by mathematical induction or by a combinatorial argument.

$r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.

7. **(+)** Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

Creating Equations★ A-CED

Create equations that describe numbers or relationships.

1. Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.
3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.*
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law* $V = IR$ *to highlight resistance* R.

Reasoning with Equations and Inequalities A-REI

Understand solving equations as a process of reasoning and explain the reasoning.

1. Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.
2. Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Solve equations and inequalities in one variable.

3. Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.
4. Solve quadratic equations in one variable.
   a. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x - p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.
   b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

Solve systems of equations.

5. Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.

6. Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

7. Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. *For example, find the points of intersection between the line* $y = -3x$ *and the circle* $x^2 + y^2 = 3$.

8. **(+)** Represent a system of linear equations as a single matrix equation in a vector variable.

9. **(+)** Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension 3 × 3 or greater).

Represent and solve equations and inequalities graphically.

10. Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).

11. Explain why the *x*-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or g(x) are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.★

12. Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes.

Mathematics | High School—Functions

FUNCTIONS describe situations where one quantity determines another. For example, the return on $10,000 in-vested at an annualized percentage rate of 4.25% is a function of the length of time the money is invested. Because we continually make theories about dependencies between quantities in nature and society, functions are important tools in the construction of mathematical models.

In school mathematics, functions usually have numerical inputs and outputs and are often defined by an algebraic expression. For example, the time in hours it takes for a car to drive 100 miles is a function of the car's speed in miles per hour, v; the rule $T(v) = 100/v$ expresses this relationship algebraically and defines a function whose name is T.

The set of inputs to a function is called its domain. We often infer the domain to be all inputs for which the expression defining a function has a value, or for which the function makes sense in a given context.

A function can be described in various ways, such as by a graph (e.g., the trace of a seismograph); by a verbal rule, as in, "I'll give you a state, you give me the capital city;" by an algebraic expression like $f(x) = a + bx$; or by a recursive rule. The graph of a function is often a useful way of visualizing the relationship of the function models, and manipulating a mathematical expression for a function can throw light on the function's properties.

Functions presented as expressions can model many important phenomena. Two important families of functions characterized by laws of growth are linear functions, which grow at a constant rate, and exponential functions, which grow at a constant percent rate. Linear functions with a constant term of zero describe proportional relationships.

A graphing utility or a computer algebra system can be used to experiment with properties of these functions and their graphs and to build computational models of functions, including recursively defined functions.

Connections to Expressions, Equations, Modeling, and Coordinates. Determining an output value for a particular input involves evaluating an expression; finding inputs that yield a given output involves solving an equation. Questions about when two functions have the same value for the same input lead to equations, whose solutions can be visualized from the intersection of their graphs. Because functions describe relationships between quantities, they are frequently used in modeling. Sometimes functions are defined by a recursive process, which can be displayed effectively using a spreadsheet or other technology.

Functions Overview

Interpreting Functions

- **Understand the concept of a function and use function notation.**
- **Interpret functions that arise in applications in terms of the context.**
- **Analyze functions using different representations.**

Building Functions

- **Build a function that models a relationship between two quantities.**
- **Build new functions from existing functions.**

Linear, Quadratic, and Exponential Models

- **Construct and compare linear, quadratic, and exponential models and solve problems.**
- **Interpret expressions for functions in terms of the situation they model.**

Trigonometric Functions

- **Extend the domain of trigonometric functions using the unit circle.**
- **Model periodic phenomena with trigonometric functions.**
- **Prove and apply trigonometric identities.**

MATHEMATICAL PRACTICES

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Interpreting Functions F-IF

Understand the concept of a function and use function notation.

1. Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$.
2. Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.
3. Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *For example, the Fibonacci sequence is defined recursively by* $f(0) = f(1) = 1$, $f(n+1) = f(n) + f(n-1)$ *for* $n \geq 1$.

Interpret functions that arise in applications in terms of the context.

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★
6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★

Analyze functions using different representations.

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★
   - **a.** Graph linear and quadratic functions and show intercepts, maxima, and minima.
   - **b.** Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.
   - **c.** Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.
   - **d.** **(+)** Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior.
   - **e.** Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.
8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
   - **a.** Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.

b. Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as* $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{12t}$, $y = (1.2)^{t/10}$, *and classify them as representing exponential growth or decay.*

9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*

Building Functions F-BF

Build a function that models a relationship between two quantities.

1. Write a function that describes a relationship between two quantities.★

a. Determine an explicit expression, a recursive process, or steps for calculation from a context.

b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.*

c. **(+)** Compose functions. *For example, if T(y) is the temperature in the atmosphere as a function of height, and h(t) is the height of a weather balloon as a function of time, then T(h(t)) is the temperature at the location of the weather balloon as a function of time*.

2. Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.★

Build new functions from existing functions.

3. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

4. Find inverse functions.

a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. F*or example,* $f(x) = 2x^3$ *or* $f(x) = (x+1)/(x-1)$ *for* $x \neq 1$.

b. **(+)** Verify by composition that one function is the inverse of another.

c. **(+)** Read values of an inverse function from a graph or a table, given that the function has an inverse.

d. **(+)** Produce an invertible function from a non-invertible function by restricting the domain.

5. **(+)** Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

Linear, Quadratic, and Exponential Models★ F-LE

Construct and compare linear, quadratic, and exponential models and solve problems.

1. Distinguish between situations that can be modeled with linear functions and with exponential functions.

a. Prove that linear functions grow by equal differences over equal intervals and that exponential functions grow by equal factors over equal intervals.

b. Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.

c. Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.

2. Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).

3. Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function.

4. For exponential models, express as a logarithm the solution to $a\,b^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.

Interpret expressions for functions in terms of the situation they model.

5. Interpret the parameters in a linear, quadratic, or exponential function in terms of a context.

Trigonometric Functions F-TF

Extend the domain of trigonometric functions using the unit circle.

1. Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

2. Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

3. Use special triangles to determine geometrically the values of sine, cosine, tangent for $\pi/3$, $\pi/4$ and $\pi/6$, and use the unit circle to express the values of sine, cosine, and tangent for $\pi-x$, $\pi+x$, and $2\pi-x$ in terms of their values for x, where x is any real number.

4. **(+)** Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

Model periodic phenomena with trigonometric functions.

5. Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.★

6. **(+)** Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.

7. **(+)** Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context.★

Prove and apply trigonometric identities.

8. Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant.

9. **(+)** Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

Mathematics | High School—Modeling

MODELING links classroom mathematics and statistics to everyday life, work, and decision-making. Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions. Quantities and their relationships in physical, economic, public policy, social, and everyday situations can be modeled using mathematical and statistical methods. When making mathematical models, technology is valuable for varying assumptions, exploring consequences, and comparing predictions with data.

> A model can be very simple, such as writing total cost as a product of unit price and number bought, or using a geometric shape to describe a physical object like a coin. Even such simple models involve making choices. It is up to us whether to model a coin as a three-dimensional cylinder, or whether a two-dimensional disk works well enough for our purposes. Other situations—modeling a delivery route, a production schedule, or a comparison of loan amortizations—need more elaborate models that use other tools from the mathematical sciences. Real-world situations are not organized and labeled for analysis; formulating tractable models, representing such models, and analyzing them is appropriately a creative process. Like every such process, this depends on acquired expertise as well as creativity.

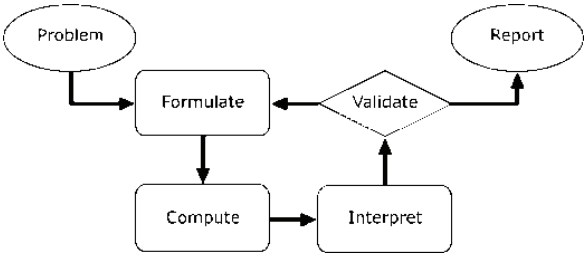
Some examples of such situations might include:

- Estimating how much water and food is needed for emergency relief in a devastated city of 3 million people, and how it might be distributed.
- Planning a table tennis tournament for 7 players at a club with 4 tables, where each player plays against each other player.
- Designing the layout of the stalls in a school fair so as to raise as much money as possible.
- Analyzing stopping distance for a car.
- Modeling savings account balance, bacterial colony growth, or investment growth.
- Engaging in critical path analysis, e.g., applied to turnaround of an aircraft at an airport.
- Analyzing risk in situations such as extreme sports, pandemics, and terrorism.
- Relating population statistics to individual predictions.

In situations like these, the models devised depend on a number of factors: How precise an answer do we want or need? What aspects of the situation do we most need to understand, control, or optimize? What resources of time and tools do we have? The range of models that we can create and analyze is also constrained by the limitations of our mathematical, statistical, and technical skills, and our ability to recognize significant variables and relationships among them. Diagrams of various kinds, spreadsheets and other technology, and algebra are powerful tools for understanding and solving problems drawn from different types of real-world situations.

One of the insights provided by mathematical modeling is that essentially the same mathematical or statistical structure can sometimes model seemingly different situations. Models can also shed light on the mathematical structures themselves, for example, as when a model of bacterial growth makes more vivid the explosive growth of the exponential function.

The basic modeling cycle is summarized in the diagram.



It involves:

(1) Identifying variables in the situation and selecting those that represent essential features.
(2) Formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables.
(3) Analyzing and performing operations on these relationships to draw conclusions.
(4) Interpreting the results of the mathematics in terms of the original situation.
(5) Validating the conclusions by comparing them with the situation, and then either improving the model or, if it is acceptable, reporting on the conclusions and the reasoning behind them. Choices, assumptions, and approximations are present throughout this cycle.

In descriptive modeling, a model simply describes the phenomena or summarizes them in a compact form. Graphs of observations are a familiar descriptive model—for example, graphs of global temperature and atmospheric CO_2 over time.

Analytic modeling seeks to explain data on the basis of deeper theoretical ideas, albeit with parameters that are empirically based; for example, exponential growth of bacterial colonies (until cut-off mechanisms such as pollution or starvation intervene) follows from a constant reproduction rate. Functions are an important tool for analyzing such problems.

Graphing utilities, spreadsheets, computer algebra systems, and dynamic geometry software are powerful tools that can be used to model purely mathematical phenomena (e.g., the behavior of polynomials) as well as physical phenomena.

MODELING STANDARDS *Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol★.*

Mathematics | High School—Geometry

An understanding of the attributes and relationships of geometric objects can be applied in diverse contexts—interpreting a schematic drawing, estimating the amount of wood needed to frame a sloping roof, rendering computer graphics, or designing a sewing pattern for the most efficient use of material.

Although there are many types of geometry, school mathematics is devoted primarily to plane Euclidean geometry, studied both synthetically (without coordinates) and analytically (with coordinates). Euclidean geometry is characterized most importantly by the Parallel Postulate, that through a point not on a given line there is exactly one parallel line. (Spherical geometry, in contrast, has no parallel lines.)

During high school, students begin to formalize their geometry experiences from elementary and middle school, using more precise definitions and developing careful proofs. Later in college some students develop Euclidean and other geometries carefully from a small set of axioms.

The concepts of congruence, similarity, and symmetry can be understood from the perspective of geometric transformation. Fundamental are the rigid motions: translations, rotations, reflections, and combinations of these, all of which are here assumed to preserve distance and angles (and therefore shapes generally). Reflections and rotations each explain a particular type of symmetry, and the symmetries of an object offer insight into its attributes—as when the reflective symmetry of an isosceles triangle assures that its base angles are congruent.

In the approach taken here, two geometric figures are defined to be congruent if there is a sequence of rigid motions that carries one onto the other. This is the principle of superposition. For triangles, congruence means the equality of all corresponding pairs of sides and all corresponding pairs of angles. During the middle grades, through experiences drawing triangles from given conditions, students notice ways to specify enough measures in a triangle to ensure that all triangles drawn with those measures are congruent. Once these triangle congruence criteria (ASA, SAS, and SSS) are established using rigid motions, they can be used to prove theorems about triangles, quadrilaterals, and other geometric figures.

Similarity transformations (rigid motions followed by dilations) define similarity in the same way that rigid motions define congruence, thereby formalizing the similarity ideas of “same shape” and “scale factor” developed in the middle grades. These transformations lead to the criterion for triangle similarity that two pairs of corresponding angles are congruent.

The definitions of sine, cosine, and tangent for acute angles are founded on right triangles and similarity, and, with the Pythagorean Theorem, are fundamental in many real-world and theoretical situations. The Pythagorean Theorem is generalized to non-right triangles by the Law of Cosines. Together, the Laws of Sines and Cosines em-body the triangle congruence criteria for the cases where three pieces

of information suffice to completely solve a triangle. Furthermore, these laws yield two possible solutions in the ambiguous case, illustrating that Side-Side-Angle is not a congruence criterion.

Analytic geometry connects algebra and geometry, resulting in powerful methods of analysis and problem solving. Just as the number line associates numbers with locations in one dimension, a pair of perpendicular axes associates pairs of numbers with locations in two dimensions. This correspondence between numerical coordinates and geometric points allows methods from algebra to be applied to geometry and vice versa. The solution set of an equation becomes a geometric curve, making visualization a tool for doing and understanding algebra. Geometric shapes can be described by equations, making algebraic manipulation into a tool for geometric understanding, modeling, and proof. Geometric transformations of the graphs of equations correspond to algebraic changes in their equations.

Dynamic geometry environments provide students with experimental and modeling tools that allow them to investigate geometric phenomena in much the same way as computer algebra systems allow them to experiment with algebraic phenomena.

CONNECTIONS TO EQUATIONS. The correspondence between numerical coordinates and geometric points allows methods from algebra to be applied to geometry and vice versa. The solution set of an equation becomes a geometric curve, making visualization a tool for doing and understanding algebra. Geometric shapes can be described by equations, making algebraic manipulation into a tool for geometric understanding, modeling, and proof.

Geometry Overview

Congruence

- **Experiment with transformations in the plane.**
- **Understand congruence in terms of rigid motions.**
- **Prove geometric theorems.**
- **Make geometric constructions.**

Similarity, Right Triangles, and Trigonometry

- **Understand similarity in terms of similarity transformations.**
- **Prove theorems involving similarity.**
- **Define trigonometric ratios and solve problems involving right triangles.**
- **Apply trigonometry to general triangles.**

Circles

- **Understand and apply theorems about circles.**
- **Find arc lengths and areas of sectors of circles.**

Expressing Geometric Properties with Equations

- **Translate between the geometric description and the equation for a conic section.**
- **Use coordinates to prove simple geometric theorems algebraically.**

Geometric Measurement and Dimension

- **Explain volume formulas and use them to solve problems.**
- **Visualize relationships between two-dimensional and three-dimensional objects.**

Modeling with Geometry

- **Apply geometric concepts in modeling situations.**

MATHEMATICAL PRACTICES

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Congruence G-CO

Experiment with transformations in the plane.

1. Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.
2. Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).
3. Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.
4. Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.
5. Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

Understand congruence in terms of rigid motions.

6. Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.
7. Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.
8. Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

Prove geometric theorems.

9. Prove theorems about lines and angles. *Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.*
10. Prove theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.*
11. Prove theorems about parallelograms. *Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.*

Make geometric constructions.

12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). *Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.*
13. Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

Similarity, Right Triangles, and Trigonometry G-SRT

Understand similarity in terms of similarity transformations.

1. Verify experimentally the properties of dilations given by a center and a scale factor:
   - **a.** A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.
   - **b.** The dilation of a line segment is longer or shorter in the ratio given by the scale factor.
2. Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.
3. Use the properties of similarity transformations to establish the AA criterion for two triangles to be similar.

Prove theorems involving similarity.

4. Prove theorems about triangles. Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.
5. Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Define trigonometric ratios and solve problems involving right triangles.

6. Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.
7. Explain and use the relationship between the sine and cosine of complementary angles.
8. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.★

Apply trigonometry to general triangles.

9. **(+)** Derive the formula A = ½ ab sin(C) for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.
10. **(+)** Prove the Laws of Sines and Cosines and use them to solve problems.
11. **(+)** Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Circles G-C

Understand and apply theorems about circles.

1. Prove that all circles are similar.
2. Identify and describe relationships among inscribed angles, radii, and chords. Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.
3. Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.
4. **(+)** Construct a tangent line from a point outside a given circle to the circle.

Find arc lengths and areas of sectors of circles.

5. Derive using similarity the fact that the length of the arc intercepted by an angle is propor-

tional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

Expressing Geometric Properties with Equations G-GPE

Translate between the geometric description and the equation for a conic section.

1. Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.
2. Derive the equation of a parabola given a focus and directrix.
3. **(+)** Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.

Use coordinates to prove simple geometric theorems algebraically.

4. Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, √3) lies on the circle centered at the origin and containing the point (0, 2).*
5. Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).
6. Find the point on a directed line segment between two given points that partitions the segment in a given ratio.
7. Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula.★

Geometric Measurement and Dimension G-GMD

Explain volume formulas and use them to solve problems.

1. Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. Use dissection arguments, Cavalieri's principle, and informal limit arguments.
2. **(+)** Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.
3. Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.★

Visualize relationships between two-dimensional and three-dimensional objects.

4. Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

Modeling with Geometry G-MG

Apply geometric concepts in modeling situations.

1. Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).★
2. Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).★
3. Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).★

Mathematics | High School—Statistics and Probability★

Decisions or predictions are often based on data—numbers in context. These decisions or predictions would be easy if the data always sent a clear message, but the message is often obscured by variability. Statistics provides tools for describing variability in data and for making informed decisions that take it into account.

Data are gathered, displayed, summarized, examined, and interpreted to discover patterns and deviations from patterns. Quantitative data can be described in terms of key characteristics: measures of shape, center, and spread. The shape of a data distribution might be described as symmetric, skewed, flat, or bell shaped, and it might be summarized by a statistic measuring center (such as mean or median) and a statistic measuring spread (such as standard deviation or interquartile range). Different distributions can be compared numerically using these statistics or compared visually using plots. Knowledge of center and spread are not enough to describe a distribution. Which statistics to compare, which plots to use, and what the results of a comparison might mean, depend on the question to be investigated and the real-life actions to be taken.

Randomization has two important uses in drawing statistical conclusions. First, collecting data from a random sample of a population makes it possible to draw valid conclusions about the whole population, taking variability into account. Second, randomly assigning individuals to different treatments allows a fair comparison of the effectiveness of those treatments. A statistically significant outcome is one that is unlikely to be due to chance alone, and this can be evaluated only under the condition of randomness. The conditions under which data are collected are important in drawing conclusions from the data; in critically reviewing uses of statistics in public media and other reports, it is important to consider the study design, how the data were gathered, and the analyses employed as well as the data summaries and the conclusions drawn.

Random processes can be described mathematically by using a probability model: a list or description of the possible outcomes (the sample space), each of which is assigned a probability. In situations such as flipping a coin, rolling a number cube, or drawing a card, it might be reasonable to assume various outcomes are equally likely. In a probability model, sample points represent outcomes and combine to make up events; probabilities of events can be computed by applying the Addition and Multiplication Rules. Interpreting these probabilities relies on an understanding of independence and conditional probability, which can be approached through the analysis of two-way tables.

Technology plays an important role in statistics and probability by making it possible to generate plots, regression functions, and correlation coefficients, and to simulate many possible outcomes in a short amount of time.

CONNECTIONS TO FUNCTIONS AND MODELING. Functions may be used to describe data; if the data suggest a linear relationship, the relationship can be modeled with a regression line, and its strength and direction can be ex-pressed through a correlation coefficient.

Statistics and Probability Overview

Interpreting Categorical and Quantitative Data

- **Summarize, represent, and interpret data on a single count or measurement variable.**
- **Summarize, represent, and interpret data on two categorical and quantitative variables.**
- **Interpret linear models.**

Making Inferences and Justifying Conclusions

- **Understand and evaluate random processes underlying statistical experiments.**
- **Make inferences and justify conclusions from sample surveys, experiments and observational studies.**

Conditional Probability and the Rules of Probability

- **Understand independence and conditional probability and use them to interpret data.**
- **Use the rules of probability to compute probabilities of compound events in a uniform probability model.**

Using Probability to Make Decisions

- **Calculate expected values and use them to solve problems.**
- **Use probability to evaluate outcomes of decisions.**

MATHEMATICAL PRACTICES

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Interpreting Categorical and Quantitative Data S-ID

Summarize, represent, and interpret data on a single count or measurement variable.

1. Represent data with plots on the real number line (dot plots, histograms, and box plots).
2. Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets.
3. Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers).
4. Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve.

Summarize, represent, and interpret data on two categorical and quantitative variables.

5. Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data.
6. Represent data on two quantitative variables on a scatter plot, and describe how the variables are related.
   - **a.** Fit a function to the data; use functions fitted to data to solve problems in the context of the data. *Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models.*
   - **b.** Informally assess the fit of a function by plotting and analyzing residuals.
   - **c.** Fit a linear function for a scatter plot that suggests a linear association.

Interpret linear models.

7. Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data.
8. Compute (using technology) and interpret the correlation coefficient of a linear fit.
9. Distinguish between correlation and causation.

Making Inferences and Justifying Conclusions S-IC

Understand and evaluate random processes underlying statistical experiments.

1. Understand statistics as a process for making inferences to be made about population parameters based on a random sample from that population.
2. Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. *For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?*

Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

3. Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each.
4. Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling.
5. Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant.
6. Evaluate reports based on data.

Conditional Probability and the Rules of Probability S-CP

Understand independence and conditional probability and use them to interpret data.

1. Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not").
2. Understand that two events *A* and *B* are independent if the probability of A and B occurring together is the product of their probabilities, and use this characterization to determine if they are independent.
3. Understand the conditional probability of *A* given *B* as *P(A and B)/P(B)*, and interpret independence of *A* and *B* as saying that the conditional probability of *A* given *B* is the same as the probability of *A*, and the conditional probability of *B* given *A* is the same as the probability of *B*.
4. Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. *For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results.*
5. Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. *For example, compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer.*

Use the rules of probability to compute probabilities of compound events in a uniform probability model.

6. Find the conditional probability of *A* given *B* as the fraction of *B*'s outcomes that also belong to *A*, and interpret the answer in terms of the model.
7. Apply the Addition Rule, *P(A or B) = P(A) + P(B) – P(A and B)*, and interpret the answer in terms of the model.
8. **(+)** Apply the general Multiplication Rule in a uniform probability model, *P(A and B) = P(A)P(B|A) = P(B)P(A|B)*, and interpret the answer in terms of the model.
9. **(+)** Use permutations and combinations to compute probabilities of compound events and solve problems.

Using Probability to Make Decisions S-MD

Calculate expected values and use them to solve problems.

1. **(+)** Define a random variable for a quantity of interest by assigning a numerical value to each event in a sample space; graph the corresponding probability distribution using the same graphical displays as for data distributions.
2. **(+)** Calculate the expected value of a random variable; interpret it as the mean of the probability distribution.
3. **(+)** Develop a probability distribution for a random variable defined for a sample space in which theoretical probabilities can be calculated; find the expected value. *For example, find the theoretical probability distribution for the number of correct answers obtained by guessing on all five questions of a multiple-choice test where each question has four choices, and find the expected grade under various grading schemes*.
4. **(+)** Develop a probability distribution for a random variable defined for a sample space in which probabilities are assigned empirically; find the expected value. *For example, find a current data distribution on the number of TV sets per household in the United States, and calculate the expected number of sets per household. How many TV sets would you expect to find in 100 randomly selected households?*

Use probability to evaluate outcomes of decisions.

5. **(+)** Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values.
   - **a.** Find the expected payoff for a game of chance. *For example, find the expected winnings from a state lottery ticket or a game at a fast-food restaurant.*
   - **b.** Evaluate and compare strategies on the basis of expected values. *For example, compare a high-deductible versus a low-deductible automobile insurance policy using various, but reasonable, chances of having a minor or a major accident.*
6. **(+)** Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).
7. **(+)** Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game).

Utah STATE OFFICE of Education

UTAH CORE STATE STANDARDS *for* MATHEMATICS SECONDARY COURSES

OVERVIEW OF THE INTEGRATED PATHWAY FOR THE COMMON CORE STATE MATHEMATICS STANDARDS

This table shows the domains and clusters in each course in the Integrated Pathway. The standards from each cluster included in that course are listed below each cluster. For each course, limits and focus for the clusters are shown in italics.

	Domains	Mathematics I	Mathematics II	Mathematics III	Fourth Courses*
Number And Quantity	**The Real Number System**		▸ Extend the properties of exponents to rational exponents. **N.RN.1, 2** ▸ Use properties of rational and irrational numbers. **N.RN.3**		* The **(+)** standards in this column are those in the Common Core State Standards that are not included in any of the Integrated Pathway courses. They would be used in additional courses developed to follow Mathematics III.
	Quantities	▸ Reason quantitatively and use units to solve problems. *Foundation for work with expressions, equations and functions* **N.Q.1, 2, 3**			
	The Complex Number System		▸ Perform arithmetic operations with complex numbers. i^2 as highest power of *i*. **N.CN.1, 2** ▸ Use complex numbers in polynomial identities and equations. *Quadratics with real coefficients* **N.CN.7, (+) 8, (+) 9**	▸ Use complex numbers in polynomial identities and equations. *Polynomials with real coefficients; apply N.CN.9 to higher degree polynomials* **(+) N.CN. 8, 9**	▸ Perform arithmetic operations with complex numbers. **(+) N.CN.3** ▸ Represent complex numbers and their operations on the complex plane. **(+) N.CN.4, 5, 6**
	Vector Quantities and Matrices				▸ Represent and model with vector quantities. **(+) N.VM.1, 2, 3** ▸ Perform operations on vectors. **(+) N.VM.4a, 4b, 4c, 5a, 5b** ▸ Perform operations on matrices and use matrices in applications. **(+) N.VM.6, 7, 8, 9, 10, 11, 12**

	Domains	Mathematics I	Mathematics II	Mathematics III	Fourth Courses*
Algebra	**Seeing Structure in Expressions**	▸ Interpret the structure of expressions. *Linear expressions and exponential expressions with integer exponents* **A.SSE.1a, 1b**	▸ Interpret the structure of expressions. *Quadratic and exponential* **A.SSE.1a, 1b, 2** ▸ Write expressions in equivalent forms to solve problems. *Quadratic and exponential* **A.SSE.3a, 3b, 3c**	▸ Interpret the structure of expressions. *Polynomial and rational* **A.SSE.1a, 1b, 2** ▸ Write expressions in equivalent forms to solve problems. **A.SSE.4**	
	Arithmetic With Polynomials and Rational Expressions		▸ Perform arithmetic operations on polynomials. *Polynomials that simplify to quadratics* **A.APR.1**	▸ Perform arithmetic operations on polynomials. *Beyond quadratic* **A.APR.1** ▸ Understand the relationship between zeros and factors of polynomials. **A.APR.2, 3** ▸ Use polynomial identities to solve problems. **A.APR.4, (+) 5** ▸ Rewrite rational expressions. *Linear and quadratic denominators* A.APR.6, (+) 7	
	Creating Equations	▸ Create equations that describe numbers or relationships. *Linear, and exponential (integer inputs only); for A.CED.3, linear only* **A.CED. 1, 2, 3, 4**	▸ Create equations that describe numbers or relationships. *In A.CED.4, include formulas involving quadratic terms* **A.CED. 1, 2, 4**	▸ Create equations that describe numbers or relationships. *Equations using all available types of expressions including simple root functions* **A.CED.1, 2, 3, 4**	

	Domains	Mathematics I	Mathematics II	Mathematics III	Fourth Courses*
Algebra	**Reasoning With Equations and Inequalities**	▸ Understand solving equations as a process of reasoning and explain the reasoning. *Master linear, learn as general principle* **A.REI.1** ▸ Solve equations and inequalities in one variable. *Linear inequalities; literal that are linear in the variables being solved for;, exponential of a form, such as $2^x = 1/16$* **A.REI.3** ▸ Solve systems of equations. *Linear systems* **A.REI.5, 6** ▸ Represent and solve equations and inequalities graphically. *Linear and exponential; learn as general principle* **A.REI.10, 11, 12**	▸ Solve equations and inequalities in one variable. *Quadratics with real coefficients* **A.REI.4a, 4b** ▸ Solve systems of equations. *Linear-quadratic systems* **A.REI.7**	▸ Understand solving equations as a process of reasoning and explain the reasoning. *Simple radical and rational* **A.REI.2** ▸ Represent and solve equations and inequalities graphically. *Combine polynomial, rational, radical, absolute value, and exponential functions* **A.REI.11**	▸ Solve systems of equations. **(+) A.REI.8, 9**
Functions	**Interpreting Functions**	▸ Understand the concept of a function and use function notation. *Learn as general principle. Focus on linear and exponential (integer domains) and on arithmetic and geometric sequences* **F.IF.1, 2, 3** ▸ Interpret functions that arise in applications in terms of a context. *Linear and exponential, (linear domain)* F.IF.4, 5, 6 ▸ Analyze functions using different representations. *Linear and exponential* **F.IF.7a, 7e, 9**	▸ Interpret functions that arise in applications in terms of a context. *Quadratic* **F.IF.4, 5, 6** ▸ Analyze functions using different representations. *Linear, exponential, quadratic, absolute value, step, piecewise- defined* **F.IF.7a, 7b, 8a, 8b, 9**	▸ Interpret functions that arise in applications in terms of a context. *Include rational, square root and cube root; emphasize selection of appropriate models* **F.IF.4, 5, 6** ▸ Analyze functions using different representations. *Include rational and radical; focus on using key features to guide selection of appropriate type of model function* **F.IF. 7b, 7c, 7e, 8, 9**	▸ Analyze functions using different representations. *Logarithmic and trigonometric functions* **(+) F.IF.7d**

	Domains	Mathematics I	Mathematics II	Mathematics III	Fourth Courses*
Functions	**Building Functions**	▸ Interpret the structure of expressions. *Linear expressions and exponential expressions with integer exponents* **A.SSE.1a, 1b**	▸ Interpret the structure of expressions. *Quadratic and exponential* **A.SSE.1a, 1b, 2** ▸ Write expressions in equivalent forms to solve problems. *Quadratic and exponential* **A.SSE.3a, 3b, 3c**	▸ Build a function that models a relationship between two quantities. *Include all types of functions studied* **F.BF.1b** ▸ Build new functions from existing functions. *Include simple radical, rational, and exponential functions; emphasize common effect of each transformation across function types* **F.BF.3, 4a**	▸ Build a function that models a relationship between two quantities. **(+) F.BF.1c** ▸ Build new functions from existing functions. **(+) F.BF.4b, 4c, 4d, 5**
	Linear, Quadratic, and Exponential Models	▸ Construct and compare linear, quadratic, and exponential models and solve problems. *Linear and exponential* **F.LE.1a, 1b, 1c, 2, 3** ▸ Interpret expressions for functions in terms of the situation they model. *Linear and exponential of form $f(x) = b^x + k$* **F.LE.5**	▸ Construct and compare linear, quadratic, and exponential models and solve problems. *Include quadratic* **F.LE. 3**	▸ Construct and compare linear, quadratic, and exponential models and solve problems. *Logarithms as solutions for exponentials* **F.LE.4**	
	Trigonometric Functions		▸ Prove and apply trigonometric identities. **F.TF.8**	▸ Extend the domain of trigonometric functions using the unit circle. **F.TF.1, 2, 3** ▸ Model periodic phenomena with trigonometric functions. **F.TF. 5**	▸ Extend the domain of trigonometric functions using the unit circle. **(+) F.TF.3, 4** ▸ Model periodic phenomena with trigonometric functions. **(+) F.TF. 6, 7** ▸ Prove and apply trigonometric identities. **(+) F.TF. 8**

	Domains	Mathematics I	Mathematics II	Mathematics III	Fourth Courses*
Geometry	**Congruence**	▸ Experiment with transformations in the plane. **G.CO.1, 2, 3, 4, 5** ▸ Understand congruence in terms of rigid motions. *Build on rigid motions as a familiar starting point for development of concept of geometric proof* **G.CO.6, 7, 8** ▸ Make geometric constructions. *Formalize and explain processes* **G.CO.12, 13**	▸ Prove geometric theorems. *Focus on validity of underlying reasoning while using variety of ways of writing proofs* **G.CO.9, 10, 11**		
	Similarity, Right Triangles, and Trigonometry		▸ Understand similarity in terms of similarity transformations. **G.SRT.1a, 1b, 2, 3** ▸ Prove theorems involving similarity. *Focus on validity of underlying reasoning while using variety of formats* **G.SRT.4, 5** ▸ Define trigonometric ratios and solve problems involving right triangles. **G.SRT.6, 7, 8**	▸ Apply trigonometry to general triangles. **(+) G.SRT.9. 10, 11**	
	Circles		▸ Understand and apply theorems about circles. **G.C.1, 2, 3, (+) 4** ▸ Find arc lengths and areas of sectors of circles. *Radian introduced only as unit of measure* **G.C.5**		

	Domains	Mathematics I	Mathematics II	Mathematics III	Fourth Courses*
Geometry	**Expressing Geometric Properties with Equations**	▸ Use coordinates to prove simple geometric theorems algebraically. *Include distance formula; relate to Pythagorean theorem* **G.GPE. 4, 5, 7**	▸ Translate between the geometric description and the equation for a conic section. **G.GPE.1, 2** ▸ Use coordinates to prove simple geometric theorems algebraically. *For G.GPE.4 in-clude simple circle theorems* **G.GPE.4**		▸ Translate between the geometric description and the equation for a conic section. **(+) G.GPE.3**
Geometry	**Geometric Measurement and Dimension**		▸ Explain volume formulas and use them to solve problems. **G.GMD.1, 3**	▸ Visualize the relation between two-dimensional and three- dimensional objects. **G.GMD.4**	▸ Explain volume formulas and use them to solve problems. **(+) G.GMD.2**
Geometry	**Modeling With Geometry**			▸ Apply geometric concepts in modeling situations. **G.MG.1, 2, 3**	
Statistics and Probability	**Interpreting Categorical and Quantitative Data**	▸ Summarize, represent, and interpret data on a single count or measurement variable. **S.ID.1, 2, 3** ▸ Summarize, represent, and interpret data on two categorical and quantitative variables. *Linear focus; discuss general principle* **S.ID.5, 6a, 6b, 6c** ▸ Interpret linear models. **S.ID.7, 8, 9**		▸ Summarize, represent, and interpret data on a single count or measurement variable. **S.ID.4**	
Statistics and Probability	**Making Inferences and Justifying Conclusions**			▸ Understand and evaluate random processes underlying statistical experiments. **S.IC.1, 2** ▸ Make inferences and justify conclusions from sample surveys, experiments and observational studies. **S.IC.3, 4, 5, 6**	

	Domains	Mathematics I	Mathematics II	Mathematics III	Fourth Courses*
Statistics and Probability	**Conditional Probability and the Rules of Probability**	▸ Understand independence and conditional probability and use them to interpret data. *Link to data from simulations or experiments* **S.CP.1, 2, 3, 4, 5** ▸ Use the rules of probability to compute probabilities of compound events in a uniform probability model. **S.CP.6, 7, (+) 8, (+) 9**			
	Using Probability to Make Decisions		▸ Use probability to evaluate outcomes of decisions. *Introductory; apply counting rules* **(+) S.MD.6, 7**	▸ Use probability to evaluate outcomes of decisions. *Include more complex situations* **(+) S.MD.6, 7**	▸ Calculate expected values and use them to solve problems. **(+) S.MD.1, 2, 3, 4** ▸ Use probability to evaluate outcomes of decisions. **(+) S.MD. 5a, 5b**

SECONDARY MATHEMATICS III

IT IS IN SECONDARY MATHEMATICS III that students pull together and apply the accumulation of learning that they have from their previous courses, with content grouped into four critical areas, organized into units. They apply methods from probability and statistics to draw inferences and conclusions from data. Students expand their repertoire of functions to include polynomial, rational, and radical functions.[3] They expand their study of right triangle trigonometry to include general triangles. And, finally, students bring together all of their experience with functions and geometry to create models and solve contextual problems. The Mathematical Practice Standards apply throughout each course and, together with the content standards, prescribe that students experience mathematics as a coherent, useful, and logical subject that makes use of their ability to make sense of problem situations.

CRITICAL AREA 1: In this unit, students see how the visual displays and summary statistics they learned in earlier grades relate to different types of data and to probability distributions. They identify different ways of collecting data— including sample surveys, experiments, and simulations—and the role that randomness and careful design play in the conclusions that can be drawn.

CRITICAL AREA 2: This unit develops the structural similarities between the system of polynomials and the system of integers. Students draw on analogies between polynomial arithmetic and base-ten computation, focusing on properties of operations, particularly the distributive property. Students connect multiplication of polynomials with multiplication of multi-digit integers, and division of polynomials with long division of integers. Students identify zeros of polynomials and make connections between zeros of polynomials and solutions of polynomial equations. The unit culminates with the fundamental theorem of algebra. Rational numbers extend the arithmetic of integers by allowing division by all numbers except 0. Similarly, rational expressions extend the arithmetic of polynomials by allowing division by all polynomials except the zero polynomial. A central theme of this unit is that the arithmetic of rational expressions is governed by the same rules as the arithmetic of rational numbers.

CRITICAL AREA 3: Students develop the Laws of Sines and Cosines in order to find missing measures of general (not necessarily right) triangles. They are able to distinguish whether three given measures (angles or sides) define 0, 1, 2, or infinitely many triangles. This discussion of general triangles open up the idea of trigonometry applied beyond the right triangle—that is, at least to obtuse angles. Students build on this idea to develop the notion of radian measure for angles and extend the domain of the trigonometric functions to all real numbers. They apply this knowledge to model simple periodic phenomena.

CRITICAL AREA 4: In this unit students synthesize and generalize what they have learned about a variety of function families. They extend their work with exponential functions to include solving exponential equations with logarithms. They explore the effects of transformations on graphs of diverse functions, including functions arising in an application, in order to abstract the general principle that transformations on a graph always have the same effect regardless of the type of the underlying functions. They identify appropriate types of functions to model a situation, they adjust parameters to improve the model, and they compare models by analyzing appropriateness of fit and making judgments about the domain over which a model is a good fit. The description of modeling as "the process of choosing and using mathematics and statistics to analyze empirical situations, to understand them better, and to make decisions" is at the heart of this unit. The narrative discussion and diagram of the modeling cycle should be considered when knowledge of functions, statistics, and geometry is applied in a modeling context.

[3] In this course, rational functions are limited to those whose numerators are of degree at most 1 and denominators of degree at most 2; radical functions are limited to square roots or cube roots of at most quadratic polynomials.

SECONDARY MATHEMATICS III

Units	Includes Standard Clusters*	Mathematical Practice Standards
UNIT 1 **Inferences and Conclusions From Data**	• Summarize, represent, and interpret data on single count or measurement variable. • Understand and evaluate random processes underlying statistical experiments. • Make inferences and justify conclusions from sample surveys, experiments, and observational studies. • Use probability to evaluate outcomes of decisions.	**Make sense of problems and persevere in solving them.** **Reason abstractly and quantitatively.** **Construct viable arguments and critique the reasoning of others.** **Model with mathematics.** **Use appropriate tools strategically.** **Attend to precision.** **Look for and make use of structure.** **Look for and express regularity in repeated reasoning.**
UNIT 2 **Polynomial, Rational, and Radical Relationships**	• Use complex numbers in polynomial identities and equations. • Interpret the structure of expressions. • Write expressions in equivalent forms to solve problems. • Perform arithmetic operations on polynomials. • Understand the relationship between zeros and factors of polynomials. • Use polynomial identities to solve problems. • Rewrite rational expressions. • Understand solving equations as a process of reasoning and explain the reasoning. • Represent and solve equations and inequalities graphically. • Analyze functions using different representations.	
UNIT 3† **Trigonometry of General Triangles and Trigonometric Functions**	• Apply trigonometry to general triangles. • Extend the domain of trigonometric functions using the unit circle. • Model periodic phenomena with trigonometric function.	
UNIT 4 **Mathematical Modeling**	• Create equations that describe numbers or relationships. • Interpret functions that arise in applications in terms of a context. • Analyze functions using different representations. • Build a function that models a relationship between two quantities. • Build new functions from existing functions. • Construct and compare linear, quadratic, and exponential models and solve problems. • Visualize relationships between two-dimensional and three-dimensional objects. • Apply geometric concepts in modeling situations.	

* In some cases clusters appear in more than one unit within a course or in more than one course. Instructional notes will indicate how these standards grow over time. In some cases only certain standards within a cluster are included in a unit.

UNIT 1: INFERENCES AND CONCLUSIONS FROM DATA

In this unit, students see how the visual displays and summary statistics they learned in earlier grades relate to different types of data and to probability distributions. They identify different ways of collecting data—including sample surveys, experiments, and simulations—and the role that randomness and careful design play in the conclusions that can be drawn.

UNIT 1: INFERENCES AND CONCLUSIONS FROM DATA	
Clusters With Instructional Notes	**Utah Core State Standards**
• **Summarize, represent, and interpret data on a single count or measurement variable.** While students may have heard of the normal distribution, it is unlikely that they will have prior experience using it to make specific estimates. Build on students' understanding of data distributions to help them see how the normal distribution uses area to make estimates of frequencies (which can be expressed as probabilities). Emphasize that only some data are well described by a normal distribution.	**S.ID.4** Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve.
• **Understand and evaluate random processes underlying statistical experiments.** For S.IC.2, include comparing theoretical and empirical results to evaluate the effectiveness of a treatment.	**S.IC.1** Understand that statistics allows inferences to be made about population parameters based on a random sample from that population. **S.IC.2** Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. *For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?*

UNIT 1: INFERENCES AND CONCLUSIONS FROM DATA	
Clusters With Instructional Notes	**Utah Core State Standards**
• **Make inferences and justify conclusions from sample surveys, experiments, and observational studies.** In earlier grades, students are introduced to different ways of collecting data and use graphical displays and summary statistics to make comparisons., These ideas are revisited with a focus on how the way in which data is collected determines the scope and nature of the conclusions that can be drawn from that data. The concept of statistical significance is developed informally through simulation as meaning a result that is unlikely to have occurred solely as a result of random selection in sampling or random assignment in an experiment. For S.IC.4 and 5, focus on the variability of results from experiments—that is, focus on statistics as a way of dealing with, not eliminating, inherent randomness.	**S.IC.3** Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each. **S.IC.4** Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling. **S.IC.5** Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant. **S.IC.6** Evaluate reports based on data.
• **Use probability to evaluate outcomes of decisions.** Extend to more complex probability models. Include situations such as those involving quality control or diagnostic tests that yields both false positive and false negative results.	**S.MD.6 (+)** Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator). **S.MD.7 (+)** Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game).

UNIT 2: POLYNOMIALS, RATIONAL, AND RADICAL RELATIONSHIPS

This unit develops the structural similarities between the system of polynomials and the system of integers. Students draw on analogies between polynomial arithmetic and base-ten computation, focusing on properties of operations, particularly the distributive property. Students connect multiplication of polynomials with multiplication of multidigit integers, and division of polynomials with long division of integers. Students identify zeros of polynomials and make connections between zeros of polynomials and solutions of polynomial equations. The unit culminates with the fundamental theorem of algebra. Rational numbers extend the arithmetic of integers by allowing division by all numbers except 0. Similarly, rational expressions extend the arithmetic of polynomials by allowing division by all polynomials except the zero polynomial. A central theme of this unit is that the arithmetic of rational expressions is governed by the same rules as the arithmetic of rational numbers.

UNIT 2: POLYNOMIALS, RATIONAL, AND RADICAL RELATIONSHIPS	
Clusters With Instructional Notes	**Utah Core State Standards**
• **Use complex numbers in polynomial identities and equations.** Build on work with quadratics equations in Secondary Mathematics II. Limit to polynomials with real coefficients.	**N.CN.8 (+)** Extend polynomial identities to the complex numbers. *For example, rewrite* $x^2 + 4$ *as* $(x + 2i)(x - 2i)$. **N.CN.9 (+)** Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.
• **Interpret the structure of expressions.** Extend to polynomial and rational expressions.	**A.SSE.1** Interpret expressions that represent a quantity in terms of its context.★ **a.** Interpret parts of an expression, such as terms, factors, and coefficients. **b.** Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1+r)^n$ *as the product of P and a factor not depending on P.* **A.SSE.2** Use the structure of an expression to identify ways to rewrite it. For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.
• **Write expressions in equivalent forms to solve problems.** Consider extending A.SSE.4 to infinite geometric series in curricular implementations of this course description.	**A.SSE.4** Derive the formula for the sum of a geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.*★
• **Perform arithmetic operations on polynomials.** Extend beyond the quadratic polynomials found in Secondary Mathematics II.	**A.APR.1** Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.
• **Understand the relationship between zeros and factors of polynomials.**	**A.APR.2** Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$. **A.APR.3** Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

UNIT 2: POLYNOMIALS, RATIONAL, AND RADICAL RELATIONSHIPS	
Clusters With Instructional Notes	**Utah Core State Standards**
• **Use polynomial identities to solve problems.** This cluster has many possibilities for optional enrichment, such as relating the example in A.APR.4 to the solution of the system $u^2+v^2=1$, $v = t(u+1)$, relating the Pascal triangle property of binomial coefficients to $(x+y)^{n+1} = (x+y)(x+y)^n$, deriving explicit formulas for the coefficients, or proving the binomial theorem by induction.	**A.APR.4** Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity* $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ *can be used to generate Pythagorean triples.* **A.APR.5 (+)** Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer *n*, where *x* and *y* are any numbers, with coefficients determined for example by Pascal's Triangle.
• **Rewrite rational expressions.** The limitations on rational functions apply to the rational expressions in A.APR.6. A.APR.7 requires the genera division algorithm for polynomials.	**A.APR.6** Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x) + r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system. **A.APR.7 (+)** Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.
• **Understand solving equations as a process of reasoning and explain the reasoning.** Extend to simple rational and radical equations.	**A.REI.2** Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.
• **Represent and solve equations and inequalities graphically.** Include combinations of linear, polynomial, rational, radical, absolute value, and exponential functions.	**A.REI.11** Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ *and* $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or g(x) are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.★
• **Analyze functions using different representations.** Relate F.IF.7c to the relationship between zeros of quadratic functions and their factored forms.	**F.IF.7** Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★ **c.** Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.

UNIT 3: TRIGONOMETRY OF GENERAL TRIANGLES AND TRIGONOMETRIC FUNCTIONS

Students develop the Laws of Sines and Cosines in order to find missing measures of general (not necessarily right) triangles. They are able to distinguish whether three given measures (angles or sides) define 0, 1, 2, or infinitely many triangles. This discussion of general triangles open up the idea of trigonometry applied beyond the right triangle—that is, at least to obtuse angles. Students build on this idea to develop the notion of radian measure for angles and extend the domain of the trigonometric functions to all real numbers. They apply this knowledge to model simple periodic phenomena.

UNIT 3: TRIGONOMETRY OF GENERAL TRIANGLES AND TRIGONOMETRIC FUNCTIONS	
Clusters With Instructional Notes	**Utah Core State Standards**
• **Apply trigonometry to general triangles.** With respect to the general case of the Laws of Sines and Cosines, the definitions of sine and cosine must be extended to obtuse angles.	**G.SRT.9 (+)** Derive the formula $A = ½\ ab\ sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side. **G.SRT.10 (+)** Prove the Laws of Sines and Cosines and use them to solve problems. **G.SRT.11 (+)** Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).
• **Extend the domain of trigonometric functions using the unit circle.**	**F.TF.1** Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle. **F.TF.2** Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle. **FTF3.** Use special triangles to determine geometrically the values of sine, cosine, tangent for $\pi/3$, $\pi/4$ and $\pi 6$, and use the unit circle to express the values of sine, cosine, and tangent for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of their values for x, where x is any real number.
• **Model periodic phenomena with trigonometric functions.**	**F.TF.5** Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.★

UNIT 4: MATHEMATICAL MODELING

In this unit students synthesize and generalize what they have learned about a variety of function families. They extend their work with exponential functions to include solving exponential equations with logarithms. They explore the effects of transformations on graphs of diverse functions, including functions arising in an application, in order to abstract the general principle that transformations on a graph always have the same effect regardless of the type of the underlying functions. They identify appropriate types of functions to model a situation, they adjust parameters to improve the model, and they compare models by analyzing appropriateness of fit and making judgments about the domain over which a model is a good fit. The description of modeling as *"the process of choosing and using mathematics and statistics to analyze empirical situations, to understand them better, and to make decisions"* is at the heart of this unit. The narrative discussion and diagram of the modeling cycle should be considered when knowledge of functions, statistics, and geometry is applied in a modeling context.

UNIT 4: MATHEMATICAL MODELING	
Clusters With Instructional Notes	**Utah Core State Standards**
• **Create equations that describe numbers or relationships.** For A.CED.1, use all available types of functions to create such equations, including root functions, but constrain to simple cases. While functions used in A.CED.2, 3, and 4 will often be linear, exponential, or quadratic the types of problems should draw from more complex situations than those addressed in Secondary Mathematics I. For example, finding the equation of a line through a given point perpendicular to another line allows one to find the distance from a point to a line. Note that the example given for A.CED.4 applies to earlier instances of this standard, not to the current course.	**A.CED.1** Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.* **A.CED.2** Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. **A.CED.3** Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.* **A.CED.4** Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law V = IR to highlight resistance R.*
• **Interpret functions that arise in applications in terms of a context.** Emphasize the selection of a model function based on behavior of data and context.	**F.IF.4** For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★ **F.IF.5** Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★ **F.IF.6** Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★

UNIT 4: MATHEMATICAL MODELING	
Clusters With Instructional Notes	**Utah Core State Standards**
• **Analyze functions using different representations.** Focus on applications and how key features relate to characteristics of a situation, making selection of a particular type of function model appropriate.	**F.IF.7** Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★ **b.** Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. **e.** Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. **F.IF.8** Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. **F.IF.9** Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*
• **Build a function that models a relationship between two quantities.** Develop models for more complex or sophisticated situations than in previous courses.	**F.BF.1** Write a function that describes a relationship between two quantities.★ **b.** Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.*
• **Build new functions from existing functions.** Use transformations of functions to find more optimum models as students consider increasingly more complex situations. For F.BF.3, note the effect of multiple transformations on a single function and the common effect of each transformation across function types. Include functions defined only by a graph. Extend F.BF.4a to simple rational, simple radical, and simple exponential functions; connect F.BF.4a to F.LE.4.	**F.BF.3** Identify the effect on the graph of replacing *f(x) by f(x) + k, k f(x), f(kx), and f(x + k)* for specific values of *k* (both positive and negative); find the value of *k* given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.* **F.BF.4** Find inverse functions. **a.** Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. *For example,* $f(x) = 2x^3$ *or* $f(x) = (x+1)/(x-1)$ *for* $x \neq 1$.

UNIT 4: MATHEMATICAL MODELING	
Clusters With Instructional Notes	**Utah Core State Standards**
• **Construct and compare linear, quadratic, and exponential models and solve problems.** Consider extending this unit to include the relationship between properties of logarithms and properties of exponents, such as the connection between the properties of exponents and the basic logarithm property that log xy = log x + log y.	**F.LE.4** For exponential models, express as a logarithm the solution to a $b^{ct} = d$ where *a, c*, and *d* are numbers and the base *b* is 2, 10, or *e*; evaluate the logarithm using technology.
• **Visualize relationships between two dimensional and three-dimensional objects.**	**G.GMD.4** Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.
• **Apply geometric concepts in modeling situations.**	**G.MG.1** Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).★ **G.MG.2** Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).★ **G.MG.3** Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).★

Utah STATE OFFICE of Education

UTAH CORE STATE STANDARDS *for* MATHEMATICS

TABLES

TABLE 1. Common addition and subtraction situations.[34]

	Result Unknown	Change Unknown	Start Unknown
ADD TO	Two bunnies sat on the grass. Three more bunnies hopped there. How many bunnies are on the grass now? **2 + 3 = ?**	Two bunnies were sitting on the grass. Some more bunnies hopped there. Then there were five bunnies. How many bunnies hopped over to the first two? **2 + ? = 5**	Some bunnies were sitting on the grass. Three more bunnies hopped there. Then there were five bunnies. How many bunnies were on the grass before? **? + 3 = 5**
TAKE FROM	Five apples were on the table. I ate two apples. How many apples are on the table now? **5 – 2 = ?**	Five apples were on the table. I ate some apples. Then there were three apples. How many apples did I eat? **5 – ? = 3**	Some apples were on the table. I ate two apples. Then there were three apples. How many apples were on the table before? **? – 2 = 3**

	Total Unknown	Addend Unknown	Both Addends Unknown[35]
PUT TOGETHER/ TAKE APART[36]	Three red apples and two green apples are on the table. How many apples are on the table? **3 + 2 = ?**	Five apples are on the table. Three are red and the rest are green. How many apples are green? **3 + ? = 5, 5 – 3 = ?**	Grandma has five flowers. How many can she put in her red vase and how many in her blue vase? **5 = 0 + 5, 5 = 5 + 0** **5 = 1 + 4, 5 = 4 + 1** **5 = 2 + 3, 5 = 3 + 2**

	Difference Unknown	Bigger Unknown	Smaller Unknown
COMPARE[37]	***("How many more?" version):*** Lucy has two apples. Julie has five apples. How many more apples does Julie have than Lucy? ***("How many fewer?" version):*** Lucy has two apples. Julie has five apples. How many fewer apples does Lucy have than Julie? **2 + ? = 5, 5 – 2 = ?**	***(Version with "more"):*** Julie has three more apples than Lucy. Lucy has two apples. How many apples does Julie have? ***(Version with "fewer"):*** Lucy has 3 fewer apples than Julie. Lucy has two apples. How many apples does Julie have? **2 + 3 = ?, 3 + 2 = ?**	***(Version with "more"):*** Julie has three more apples than Lucy. Julie has five apples. How many apples does Lucy have? ***(Version with "fewer"):*** Lucy has 3 fewer apples than Julie. Julie has five apples. How many apples does Lucy have? **5 – 3 = ?, ? + 3 = 5**

[34] Adapted from Box 2-4 of "Mathematics Learning in Early Childhood," National Research Council (2009, pp. 32, 33).
[35] These *take apart* situations can be used to show all the decompositions of a given number. The associated equations, which have the total on the left of the equal sign, help children understand that the = sign does not always mean *makes* or *results in* but always does mean *is the same number as*.
[36] Either addend can be unknown, so there are three variations of these problem situations. Both Addends Unknown is a productive extension of this basic situation, especially for small numbers less than or equal to 10.

TABLE 2. Common multiplication and division situations. [38]

	Unknown Product	Group Size Unknown (*"How many in each group?" Division*)	Number of Groups Unknown (*"How many groups?" Division*)
	3 × 6 = ?	**3 × ? = 18 and 18 ÷ 3 = ?**	**? × 6 = 18 and 18 ÷ 6 = ?**
EQUAL GROUPS	There are 3 bags with 6 plums in each bag. How many plums are there in all? ***Measurement example.*** You need 3 lengths of string, each 6 inches long. How much string will you need altogether?	If 18 plums are shared equally into 3 bags, then how many plums will be in each bag? ***Measurement example.*** You have 18 inches of string, which you will cut into 3 equal pieces. How long will each piece of string be?	If 18 plums are to be packed 6 to a bag, then how many bags are needed? ***Measurement example.*** You have 18 inches of string, which you will cut into pieces that are 6 inches long. How many pieces of string will you have?
ARRAYS,[39] AREA[40]	There are 3 rows of apples with 6 apples in each row. How many apples are there? ***Area example.*** What is the area of a 3 cm by 6 cm rectangle?	If 18 apples are arranged into 3 equal rows, how many apples will be in each row? ***Area example.*** A rectangle has area 18 square centimeters. If one side is 3 cm long, how long is a side next to it?	If 18 apples are arranged into equal rows of 6 apples, how many rows will there be? ***Area example.*** A rectangle has area 18 square centimeters. If one side is 6 cm long, how long is a side next to it?
COMPARE	A blue hat costs $6. A red hat costs 3 times as much as the blue hat. How much does the red hat cost? ***Measurement example.*** A rubber band is 6 cm long. How long will the rubber band be when it is stretched to be 3 times as long?	A red hat costs $18 and that is 3 times as much as a blue hat costs. How much does a blue hat cost? ***Measurement example.*** A rubber band is stretched to be 18 cm long and that is 3 times as long as it was at first. How long was the rubber band at first?	A red hat costs $18 and a blue hat costs $6. How many times as much does the red hat cost as the blue hat? ***Measurement example.*** A rubber band was 6 cm long at first. Now it is stretched to be 18 cm long. How many times as long is the rubber band now as it was at first?
GENERAL	$a \times b = ?$	$a \times ? = p$ and $p \div a = ?$	$? \times b = p$ and $p \div b = ?$

[37] For the Bigger Unknown or Smaller Unknown situations, one version directs the correct operation (the version using more for the bigger unknown and using less for the smaller unknown). The other versions are more difficult.
[38] The first examples in each cell are examples of discrete things. These are easier for students and should be given before the measurement examples.
[39] The language in the array examples shows the easiest form of array problems. A harder form is to use the terms rows and columns: The apples in the grocery window are in 3 rows and 6 columns. How many apples are in there? Both forms are valuable.
[40] Area involves arrays of squares that have been pushed together so that there are no gaps or overlaps, so array problems include these especially important measurement situations.

TABLE 3. The properties of operations. Here *a, b* and *c* stand for arbitrary numbers in a given number system. The properties of operations apply to the rational number system, the real number system, and the complex number system.

Associative property of addition	$(a + b) + c = a + (b + c)$
Commutative property of addition	$a + b = b + a$
Additive identity property of 0	$a + 0 = 0 + a = a$
Existence of additive inverses	For every a there exists $-a$ so that $a + (-a) = (-a) + a = 0$.
Associative property of multiplication	$(a \times b) \times c = a \times (b \times c)$
Commutative property of multiplication	$a \times b = b \times a$
Multiplicative identity property of 1	$a \times 1 = 1\ a = a$
Existence of multiplicative inverses	For every $a \neq 0$ there exists $1/a$ so that $a \times 1/a = 1/a \times a = 1$.
Distributive property of multiplication over addition	$a \times (b + c) = a \times b + a \times c$

TABLE 4. The properties of equality. Here *a, b* and *c* stand for arbitrary numbers in the rational, real, or complex number systems.

Reflexive property of equality	$a = a$
Symmetric property of equality	*If* $a = b$, *then* $b = a$.
Transitive property of equality	*If* $a = b$ *and* $b = c$, *then* $a = c$.
Addition property of equality	*If* $a = b$, *then* $a + c = b + c$.
Subtraction property of equality	*If* $a = b$, *then* $a - c = b - c$.
Multiplication property of equality	*If* $a = b$, *then* $a \times c = b \times c$.
Division property of equality	*If* $a = b$ *and* $c \neq 0$, *then* $a \div c = b \div c$.
Substitution property of equality	*If* $a = b$, *then* b *may be substituted for* a *in any expression containing* a.

TABLE 5. The properties of inequality. Here *a, b* and *c* stand for arbitrary numbers in the rational or real number systems.

Exactly one of the following is true: $a < b$, $a = b$, $a > b$.
If $a > b$ *and* $b > c$ *then* $a > c$.
If $a > b$, *then* $b < a$.
If $a > b$, *then* $-a < -b$.
If $a > b$, *then* $a \pm c > b \pm c$.
If $a > b$ *and* $c > 0$, *then* $a \times c > b \times c$.
If $a > b$ *and* $c < 0$, *then* $a \times c < b \times c$.
If $a > b$ *and* $c > 0$, *then* $a \div c > b \div c$.
If $a > b$ *and* $c < 0$, *then* $a \div c < b \div c$.

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SAMPLE OF WORKS CONSULTED

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